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Partitions of Points into Simplices with k-dimensional Intersection. Part II: Proof of Reay's Conjecture in Dimensions 4 and 5

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A longstanding conjecture of Reay asserts that every set X of $(m-1)(d+1)+k+1$ points in general position in $\mathbb{R}^d$ has a partition $X_1, X_2, \ldots, X_m$ such that $\bigcap_{i=1}^m \operatorname{conv} X_i$ is at least k-dimensional. Using the tools developed in [13] and oriented matroid theory, we prove this conjecture for $d = 4$ and $d = 5$. How about, to that end, we introduce the notion of a k-lopsided oriented matroid and we characterize these combinatorial objects for certain values of k. Divisibility properties for subsets of $\mathbb{R}^d$ with other independence conditions are also obtained, thus settling several particular cases of a generalization of Reay's conjecture.

1. INTRODUCTION

Following the usual terminology [10], a subset X of $\mathbb{R}^d$ is said to be (m, k)-*divisible* (for $m \geq 1$ and $0 \leq k \leq d$) if there is a partition $X_1, X_2, \ldots, X_m$ of X such that $\bigcup_{i=1}^m \operatorname{conv} X_i$ is at least k-dimensional. $(m, 0)$-divisibility just means non-empty intersection and, in this case, any point of $\bigcup_{i=1}^m \operatorname{conv} X_i$ is called a *Tverberg point* of X, in reference to Tverberg's 1966 theorem.

THEOREM 1.1 (TVERBERG'S THEOREM [14, 15]). *Let X be a set of $(m-1)(d+1)+1$ points in $\mathbb{R}^d$. Then there is a partition $X_1, X_2, \ldots, X_m$ of X such that $\bigcup_{i=1}^m \operatorname{conv} X_i \neq \phi$.*

In a recent paper, Tverberg established a similar result, concerning $(m, 1)$-divisibility.

THEOREM 1.2 ([16]). *Any set of $2d(m-1)+2$ points in $\mathbb{R}^d$ is $(m, 1)$-divisible.*

The bounds $(m-1)(d+1)+1$ in Theorem 1.1 and $2d(m-1)+2$ in Theorem 1.2 are best possible. In the first part of this paper [13], we have given new proofs of these two theorems and characterized the sets of $\mathbb{R}^d$ with $2d(m-1)+1$ elements which are not $(m, 1)$-divisible. For $k \geq 2$, a cardinality condition on $|X|$ is no longer sufficient for (m, k)-divisibility. Indeed, an independence condition is also required, to prevent, for instance, all of the points to be on a line. The simplest such condition is *general position*, which means that no $d+1$ points (or less) are affinely dependent. We will refer to Reay's conjecture as to the most appealing version of the problem of (m, k)-divisibility, still unsolved in general.

CONJECTURE 1.3 (REAY'S CONJECTURE [10]). Any set of $(m-1)(d+1)+k+1$ points in general position in $\mathbb{R}^d$ is (m, k)-divisible.

In [11], Reay defined another notion of independence, which provides a natural common extension of general position ($k = d$) and distinct points ($k = 1$). For $1 \leq k \leq d$, a subset X of $\mathbb{R}^d$ is k-*independent* if no $k+1$ elements of X (or less) are affinely dependent. Reay then proposed the following generalization of Conjecture 1.3, that Theorem 1.2 actually settles in the particular case $k = 1$.

CONJECTURE 1.4 ([11]). Any k-independent set of $(m-1)\cdot(2d-k+1)+k+1$ points in $\mathbb{R}^d$ is (m, k)-divisible.

The main goal of this paper is to prove Reay's conjecture in $\mathbb{R}^4$ and $\mathbb{R}^5$ (Theorems 7.1 and 8.1), and Conjecture 1.4 for $k \leq 3$ (Theorem 6.1). In fact, our results go further, from two

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different points of view. As for $d = 2$ [4], $d = 3$ [12] and $m = 3$ or 4 [13], we show that, in each case, the partition $X_1, X_2, \dots, X_m$ can be chosen so that $\bigcap_{i=1}^{m} \operatorname{conv} X_i$ contains a prescribed Tverberg point ω of X (we shall say, to simplify, that X is *(m, k)-divisible at ω*). On the other hand, Eckhoff noted that k-independence is somehow too restrictive for the problem of (m, k)-divisibility, and asked for a more suitable notion [8]. To that aim, we introduce the condition of k-positive independence at ω, which only concerns the properties of the convex sets containing a given point ω. Theorems 6.1, 7.1 and 8.1 are actually stated in this more general context.

The proofs of these results require several tools and preliminaries, of geometric or combinatorial nature. In each case, we proceed by finite induction, based on divisibility lemmas derived from the conic Tverberg's theorem (see [13, Theorem 2.1]). Oriented matroids [5] are also intensively used, and appear as a suitable and natural tool (more adapted, we believe, than vertex figures or Gale diagrams, which could constitute an alternative for certain lemmas). The notion of k-positive independence, which answers Eckhoff's question, has actually a simple formulation in the language of oriented matroids and leads to an extremal problem (Conjecture 4.1) that may be of interest in itself. This problem will be referred to as the '*lopsidedness conjecture*' since a positive answer would imply that the critical sets of points with respect to (m, k)-divisibility at ω are, in a certain sense, lopsided. Various examples of '*k-lopsided oriented matroids*' are described, using geometric and graph-theoretical representations, and the lopsidedness conjecture is shown to be true in several particular cases (Theorems 5.1 and 5.2). Finally, we mention that divisibility properties are also closely related to the well-known notion of *positive base* [2, 7, 9], which proves useful in certain statements. In the last sections of this paper, all of these results finally meet to establish the desired divisibility properties.

2. Oriented Matroids

We assume that the reader is familiar with the theory of oriented matroids. This section only gathers some definitions or notations which may not be classical (see [5] for the basic notions concerning this theory).

Let M be an oriented matroid on E. We recall that M is *totally cyclic* if every element of E belongs to a positive circuit, which is equivalent to saying that M^* is acyclic, where M^* denotes the dual of M. If every element of E is either a coloop or belongs to a positive circuit, we say that M is *cyclic*. Loops and coloops are generally eliminated in the reasonings, but we prefer to keep them in our definitions since they will not be a big source of complication, and will occur naturally in some examples described in Section 4. We denote by $\mathbb{A}\mathrm{ff}\,(E)$ the oriented matroid of affine dependences of a (finite) set E of $\mathbb{R}^d$. If M is *realizable* (i.e., representable over the reals) and acyclic of rank $d+1$, then M is isomorphic to such an oriented matroid. In many instances, we shall start from a subset E of $\mathbb{R}^d$ and consider $M := \mathbb{A}\mathrm{ff}\,(E \cup \omega)/\omega$, where ω is a given point of $\mathbb{R}^d$. Then M is totally cyclic if and only if ω belongs to the relative interior of conv E. In that situation, M^* can be represented as $\mathbb{A}\mathrm{ff}\,(\tilde{E})$, where $\tilde{E}$ is a finite subset of $\mathbb{R}^{|E|-d'}$, and $d' = \dim \operatorname{aff} E$. To avoid any confusion (especially when convexity properties of $\tilde{E}$ are also considered), we shall always use the notation x (resp. A) for an element (resp. a subset) of E and $\tilde{x}$ (resp. $\tilde{A}$) for the corresponding element (resp. subset) of $\tilde{E}$.

In the usual terminology of oriented matroids, a *tope* stands for a *maximal covector* [5, Section 3]. We recall that the topes of an oriented matroid M on E all have the same support, namely $E \backslash L$, where L is the set of loops of M. In the particular case where M is defined as $\mathbb{A}\mathrm{ff}\,(E \cup \omega)/\omega$ (with $\omega \notin E \subseteq \mathbb{R}^d$), M has no loops and a partition (E^+, E^-) of E is the signature of a tope of M if and only if there is a hyperplane H of $\mathbb{R}^d$ passing through ω such that $E^+ = E \cap H^+$ and $E^- = E \cap H^-$, where H^+ and H^- denote the open half-spaces defined by H. In order to keep this geometric interpretation in mind, we shall still refer to *half-spaces* of M for sets E^+ and E^- whenever (E^+, E^-) is the signature of a tope of an (arbitrary) oriented matroid M. Note that topes correspond (for loopless oriented matroids) to the more telling—but less used—notion of *non-Radon partitions* (see [2, 3]). In order to prove the existence of a half-space with 'few elements', we shall sometimes use the following lemma, which appears in an equivalent form in [1] (see also [5, Exercise 3.34]).

LEMMA 2.1 ([1]). *Let M be an oriented matroid on E and let A be a subset of E such that A and $E\backslash A$ intersect every positive circuit of M. Then M has a half-space E^- such that $E^- \subseteq A$.*

3. k-POSITIVE INDEPENDENCE AND POSITIVE BASES

A (finite) multiset X of $\mathbb{R}^d$ is said to be *k-independent* ($1 \leq k \leq d$) if no $k+1$ elements of X (or less) are affinely dependent. This definition has been introduced by Reay [11] as a natural common extension of the notions of general position ($k = d$) and distinct points ($k = 1$). However, Eckhoff pointed out in [8] that k-independence appears to be too strong a condition if, as in divisibility properties, we only focus our attention on the convex dependences. For that reason, we shall prefer to use a weaker notion, that we call *k-positive independence at ω* (now defined for $1 \leq k \leq d+1$), and which means that for any subsets A and B of X such that $\omega \in \operatorname{conv} A \cap \operatorname{conv} B$ and $|A \cup B| \leq k+1$, we also have $\omega \in \operatorname{conv}(A \cap B)$. Equivalently, for any subset C of X such that $\omega \in \operatorname{conv} C$ and $|C| \leq k+1$, there is a unique $D \subseteq C$ with minimal support, such that $\omega \in \operatorname{conv} D$. Besides circuits of oriented matroids that will be used thoroughly from now on, inclusion-minimal sets also appear in the next notion. We recall that a *positive base* of $\mathbb{R}^d$ denotes any set of vectors of $\mathbb{R}^d$ which positively spans $\mathbb{R}^d$ and is inclusion-minimal with respect to this property (see [7] and [9]).

Again, let k and d be integers such that $1 \leq k \leq d+1$ and let M denote a rank d oriented matroid on E. We say that M is *k-positively independent* if every subset A of E such that $|A| \leq k+1$ contains at most one positive circuit. By a *positive base* of M, we mean any subset B of E inclusion-minimal with the property that, for all $x \in E\backslash B$, there is a signed circuit C of M such that $C \subseteq B \cup x$ and $C^- = \{x\}$ (see [2]). These combinatorial notions generalize of course the corresponding ones in $\mathbb{R}^d$. If we take $M = \mathbb{A}\mathrm{ff}\,(X \cup \omega)/\omega$, the minimal subsets S of X such that $\omega \in \operatorname{conv} S$ are precisely the positive circuits of M, hence M is k-positively independent if and only if X is k-positively independent at ω. Moreover, if $\omega \in \operatorname{int\,conv} X$, then M is totally cyclic and $\{x_1, x_2, \ldots, x_m\}$ is a positive base of M if and only if $\{\overrightarrow{\omega x}_1, \overrightarrow{\omega x}_2, \ldots, \overrightarrow{\omega x}_m\}$ is a positive base of $\mathbb{R}^d$.

Some of the results stated below have extensions that may be of interest in themselves, but we have only kept what will be needed for divisibility problems. The first lemma has been established by Reay in $\mathbb{R}^d$ and extended by Bienia to totally cyclic oriented matroids.

LEMMA 3.1 ([2, 9]). *Let M be a totally cyclic oriented matroid on E. Then, any positive base B of M has a partition $B = B_1 \cup B_2 \cup \cdots \cup B_p$ with $|B_1| \geq |B_2| \geq \cdots \geq |B_p| \geq 2$ such that, for each j, $1 \leq j \leq p$, $\bigcup_{i=1}^{j} B_i$ is a positive base of $M(\bigcup_{i=1}^{j} B_i)$ and rank $\bigcup_{i=1}^{j} B_i = \sum_{i=1}^{j} |B_i| - j$.*

An algorithmic construction of a positive base of M together with such a partition consists in taking for each j, a positive circuit B_j of $M/\bigcup_{i=1}^{j-1} B_i$ (or of M if $j = 1$) with maximal cardinality. The reader is referred to [2] for details.

LEMMA 3.2. *Let M be a cyclic oriented matroid on E and (E_1, E_2) be a partition of E such that each positive circuit of E is included either in E_1 or in E_2. Then, M is the direct sum of $M(E_1)$ and $M(E_2)$.*

PROOF. Suppose that there exists a circuit C such that $C \cap E_1 \neq \phi$ and $C \cap E_2 \neq \phi$ and let a $\in C \cap E_1$ with, for instance, $a \in C^+$. Using sign elimination with positive circuits of $M(E_2)$, we deduce a circuit C' such that $a \in C' \cap E_1 \subseteq C \cap E_1$ and $C'^- \cap E_2 = \phi$. We have $C'^+ \cap E_2 \neq \phi$ (for otherwise, $C' \subset C$) so we can select $b \in C'^+ \cap E_2$ and use sign elimination with positive circuits of $M(E_1)$ in order to obtain a positive circuit C'' satisfying $b \in C''$ and $C'' \cap E_1 \neq \phi$, hence contradicting the hypotheses. □

LEMMA 3.3. *Let M be a k-positively independent oriented matroid on E and let C be a positive circuit of M such that $|C| \leq k$. Then C is a closed subset of E.*

PROOF. C being a positive base of $M(C)$, the existence of an element $x \in \bar{C} \backslash C$ would lead to a positive circuit C' such that $x \in C' \subseteq C \cup x$, contradicting the k-positive independence of M. □

In the following statements, X denotes a finite subset of $\mathbb{R}^d$ and ω a point of $\mathbb{R}^d$. As an immediate corollary of Lemma 3.3, we obtain the following lemma.

LEMMA 3.4. *Suppose that X is k-positively independent at ω and let S be a subset of X such that $|S| \leq k$ and $\omega \in$ relint conv S. Then, $X \cap$ aff $S = S$.*

LEMMA 3.5. *Suppose that X is k-positively independent at ω and let S and Z be disjoint subsets of X such that $\omega \in$ relint conv S and conv $Z \cap$ aff $S \neq \phi$. Then $|S| + |Z| > k + 1$.*

PROOF. Let Z' be a minimal subset of Z such that conv $Z' \cap$ aff $S \neq \phi$ Then, $\omega \in$ relint conv $(S \cup Z')$, hence the oriented matroid $M := \mathbb{A}\mathrm{ff}\ (S \cup Z' \cup \omega)/\omega$ is totally cyclic. Moreover, S is a positive circuit of M and, if $x \in Z'$, there is a positive circuit S' such that $x \in S' \subseteq S \cup Z'$. As M is k-positively independent, we have $|S| + |Z'| > k + 1$, hence $|S| + |Z| > k + 1$. □

The next sections will require divisibility lemmas for subsets of $\mathbb{R}^d$ with independence conditions. The first one has been derived in part I from the conic Tverberg's theorem.

LEMMA 3.6 ([13, LEMMA 3.3]). *Let $0 \leq k \leq d$ and let X be a set of $(m-1)\cdot(d+1)+k+1$ points in $\mathbb{R}^d$. Suppose that there exist subsets $S_1, S_2, \dots, S_m$ of X satisfying:*

(i) $S_1, S_2, \dots, S_m$ *are pairwise disjoint;*
(ii) $N := \bigcap_{i=1}^m$ conv S_i *is at least* $(k-1)$*-dimensional;*
(iii) $S_1, S_2, \dots, S_m$ *are inclusion-minimal with respect to properties* (i) *and* (ii)*;*
(iv) aff N *and* $Y := X \backslash \bigcup_{i=1}^m S_i$ *are* weakly separated *by a hyperplane, i.e., there exists a hyperplane* H *such that conv* $Y \subseteq H^+$ *and aff* $N \subseteq H^-$*, where* H^+ *and* H^- *denote the two closed half-spaces defined by* H*;*
(v) for every $Z \subseteq Y$ *such that* conv $Z \cap$ *aff* $N \neq \phi$ *and every* i, $1 \leq i \leq m$, $|S_i| + |Z| > d + 1$.

Then, there exist pairwise disjoint subsets $Y_1, Y_2, \dots, Y_m$ *of* Y *such that* $\bigcap_{i=1}^m$ *conv* $(S_i \cup Y_i)$ *is at least k-dimensional.*

LEMMA 3.7. *Let ω be a point of $\mathbb{R}^d$ and X be a set of $(m-1)\cdot(d+1)+k+1$ points in $\mathbb{R}^d$ such that X is d-positively independent at ω. Suppose that there exist subsets $S_1, S_2, \ldots, S_m$ of X satisfying:*

(i) $S_1, S_2, \ldots, S_m$ *are pairwise disjoint;*
(ii) $N := \bigcap_{i=1}^{m} \mathrm{conv} S_i$ *is at least* $(k-1)$*-dimensional* (*with* $0 \leq k \leq d$)*, and contains* ω*;*
(iii) $S_1, S_2, \ldots, S_m$ *are inclusion-minimal with respect to properties* (*i*) *and* (*ii*)*;*
(iv) aff N *and* $Y := X \setminus \bigcup_{i=1}^{m} S_i$ *are* weakly separated *by a hyperplane.*

Then, there exist pairwise disjoint subsets $Y_1, Y_2, \ldots, Y_m$ of Y such that $\bigcap_{i=1}^{m}$ conv $(S_i \cup Y_i)$ is at least k-dimensional.

PROOF. For every $Z \subseteq Y$ such that conv $Z \cap$ aff $N \neq \phi$ and every $i \leq m$, we have $|S_i| + |Z| > d+1$ by Lemma 3.5 hence Lemma 3.6 applies. $\square$

The remainder of this section relates k-positive independence to the usual topology of $\mathbb{R}^d$. First, we note that the finiteness of X and elementary topological arguments imply that the points ω such that X is k-positively independent at ω form an open set of $\mathbb{R}^d$.

LEMMA 3.8. *If X is k-positively independent at ω, then X is k-positively independent at every point of a certain neighbourhood of ω.*

The set of convex hulls conv A, with $A \subseteq X$ and dim aff $A = d-1$, naturally dissects conv X into a d-dimensional cell-complex $\mathcal{C}$. Lemma 3.8 amounts to saying that, for any two cells F and F' of $\mathcal{C}$ such that $F \subseteq F'$ and any two points $\omega \in F$ and $\omega' \in F'$, if X is k-positively independent at ω, then X is also k-positively independent at ω'. In particular, the property 'X is k-positively independent at ω' only depends on the cell F of lower dimension to which ω belongs, defining an increasing function $F \mapsto k(F)$ for the inclusion order. Similarly, for fixed m, the property 'X is (m, j)-divisible at ω' only depends on F and $F \mapsto j(F)$ defines a decreasing function. Thus, Conjecture 1.4 reduces to a problem about the cell complex $\mathcal{C}$. The proof of several particular cases of the conjecture will require ω to be in a sufficiently general position. This is guaranteed by the following procedure, that will be referred to as the *perturbation principle*. This method, already sketched at the end of Part I ([13, Theorem 5.1]) consists, roughly speaking, in proving (m, j)-divisibility at ω by 'passing to the limits'.

LEMMA 3.9 (PERTURBATION PRINCIPLE). *Let $S_1, S_2, \ldots, S_m$ be pairwise disjoint subsets of X such that $N = \bigcap_{i=1}^{m} \mathrm{conv}\, S_i$ is $(j-1)$-dimensional and contains ω. Consider ω' close to ω and in* general position *in N, i.e., ω' belongs to N but does not belong to any cell of $\mathcal{C}$ of dimension less than $j-1$. Then, for any subset A of X, $\omega' \in$ conv A implies $\omega \in$ conv A. In particular, if (m, j)-divisibility at ω' can be proved by enlarging the S_i as in Lemma* 3.6 *or Lemma* 3.7*, then (m, j)-divisibility at ω can be obtained in the same way. Moreover, the subsets S of X with minimal support such that conv $S \cap N$ is at least $(j-1)$-dimensional and contains ω are precisely the minimal subsets of X such that $\omega' \in$ conv S, i.e., the positive circuits of the oriented matroid $\mathbb{A}\mathrm{ff}\ (X \cup \omega')/\omega'$.*

4. k-LOPSIDED ORIENTED MATROIDS

In Part I [13], a purely geometric approach to divisibility properties was sufficient for our purpose, and oriented matroids only appeared in an implicit form. However, we observed in many places (and already in the proof of the conic Tverberg's theorem) the crucial role of the minimal subsets S of X such that $\omega \in$ conv S, i.e., the positive circuits of $\mathbb{A}\mathrm{ff}\ (X \cup$

$\omega)/\omega$. In addition, we pointed out the obstruction caused by the case $|S| \leq d$, even if general position is assumed [13, Remark 3.2]. More precisely, it turns out that $|S| \leq k$ is the most serious obstruction to (m, k)-divisibility if X is k-positively independent at ω. Thus, we are led to explore the combinatorial structure of the 'critical sets' X such that X is k-positively independent at ω and $|S| \leq k$ for every positive circuit S of $\mathbb{A}\mathrm{ff}\,(X \cup \omega)/\omega$. This section and the next one deal with a conjecture on oriented matroids which, if true, would imply that these sets are necessarily 'lopsided', in the sense that there would exist a hyperplane passing through ω leaving 'few' points of X on one of its sides. It should be noted that half-spaces of $\mathbb{R}^d$ containing, on the contrary, 'many' points of X have also been considered in divisibility problems (see for instance [4, 16]). Motivated by our belief in the validity of Conjecture 4.1 below, we shall say that an oriented matroid M is k-*lopsided* if M is k-positively independent and every positive circuit of M has at most k elements.

CONJECTURE 4.1 (THE 'LOPSIDEDNESS CONJECTURE'). Let k and d be integers such that $1 \leq k \leq d+1$ and let M be a k-lopsided oriented matroid of rank d on E. Then:

(i) M has a half-space E^- such that $|E^-| \leq d + 2 - k$;
(ii) if M is cyclic, $|E| \leq d + \binom{d}{k-1}$.

In Section 5, we shall establish the conjecture for certain values of k and d (Theorems 5.1 and 5.2) and these results will then be used to prove several particular cases of Conjectures 1.3 and 1.4. The details will come later, but we already note that a lopsidedness condition enters into the hypotheses of the divisibility Lemmas 3.6 and 3.7. In order to give support to Conjecture 4.1, we first present various examples of k-lopsided oriented matroids, that we believe are significant because they occur naturally in the study of divisibility properties. The first example shows that, if true, the bounds given in the lopsidedness conjecture are best possible for all k and d.

THEOREM 4.2. *Let k and d be integers such that $1 \leq k \leq d+1$. Then there is a realizable k-lopsided oriented matroid $M_{k,d}$ of rank d on E with $|E| = d + \binom{d}{k-1}$ satisfying the following properties:*

— *every positive circuit of $M_{k,d}$ has k elements;*
— *every half-space E^- of $M_{k,d}$ satisfies $|E^-| \geq d + 2 - k$;*
— *there is a half-space E^- of $M_{k,d}$ such that $|E^-| = d + 2 - k$.*

PROOF. To define $M_{1,d}$, it suffices to take d independent elements together with a loop.

Suppose $1 < k \leq d+1$ and let $B = (e_1, e_2, \dots, e_d)$ be a basis of $\mathbb{R}^d$. We construct $M_{k,d}$ as the linear oriented matroid on the set E of $d + \binom{d}{k-1}$ vectors $e_1, e_2, \dots, e_d$ and $-e_{i_1} - e_{i_2} - \cdots - e_{i_{k-1}}$ $(1 \leq i_1 < i_2 < \cdots < i_{k-1} \leq d)$. Any positive circuit C of $M_{k,d}$ must contain an element x with a negative component. Since x is $-e_{i_1} - e_{i_2} - \cdots - e_{i_{k-1}}$ and the only vectors with positive components are those of B, the other elements of C are necessarily $e_{i_1}, e_{i_2}, \dots, e_{i_{k-1}}$. It follows that each positive circuit of $M_{k,d}$ has exactly k elements and, clearly, no $k+1$ of the given vectors can form two positive circuits of $M_{k,d}$. Next, let E^- be a half-space of $M_{k,d}$ and $p = |B \backslash E^-|$. If $p \leq k-2$, we are done. Else, E^- contains $-e_{i_1} - e_{i_2} - \cdots - e_{i_{k-1}}$ for every choice of $e_{i_1}, e_{i_2}, \dots, e_{i_{k-1}}$ in $B \backslash E^-$, hence $|E^-| \geq (d-p) + \binom{p}{k-1} \geq d + 2 - k$. Finally, $E^- = \{e_{k-1}, e_k, \dots, e_d\}$ is a half-space of $M_{k,d}$ such that $|E^-| = d + 2 - k$. □

EXAMPLE 4.3. Let $X' = \{x'_1, x'_2, \dots, x'_p\}$ and $X'' = \{x''_1, x''_2, \dots, x''_p\}$ be two disjoint sets with $p \geq 2$ and let $K_{p,p}$ denote the complete bipartite directed graph on $X := X' \cup X''$, i.e., whose arcs are $x'_i x''_j$, $1 \leq i \leq p$, $1 \leq j \leq p$. Then the cographic oriented matroid

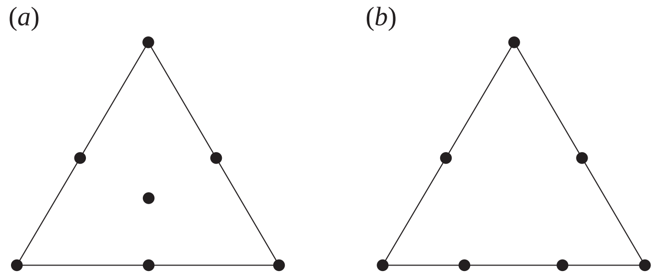

FIGURE 1.

$\mathbb{B}(K_{p,p})$ has exactly $2p$ positive circuits, each of cardinality p and the size of the union of two distinct positive circuits cannot be less than $2p-1$. Thus, $\mathbb{B}(K_{p,p})$ is totally cyclic, p-lopsided of rank $(p-1)^2$. This example can be generalized in the following way: k being a given integer such that $2 \leq k \leq p$, let us consider the directed graph on X whose arcs are $x'_i x''_i, x'_i x''_{i+1}, \dots, x'_i x''_{i+k-1}$ (with a cyclic order on the indices). Its corresponding cographic oriented matroid M is then totally cyclic, k-lopsided of rank $(k-2)p+1$. Clearly, every half-space of M has at least p elements. Conversely, applying Lemma 2.1 to the set A formed by the p arcs $x'_i x''_i$, $1 \leq i \leq p$ shows that M has a half-space with at most p elements. As $p \leq (k-2)p + 3 - k$, condition (i) of Conjecture 4.1 is satisfied. The case $k = 3$ deserves particular attention, since it yields extremal examples with respect to condition (i).

When M is totally cyclic, we observe that the facial structure of M^* retains all the necessary information for k-lopsidedness since the definition we have given only refers to positive circuits. More precisely, we have the following lemma.

LEMMA 4.4. *Let M be a totally cyclic oriented matroid of rank d on E. Then, M is k-lopsided if and only if every facet of M^* has at least $|E| - k$ elements and any two distinct facets of M^* have at most $|E| - k - 2$ elements in common.*

This geometric interpretation is now used to derive other examples of k-lopsided oriented matroids. In the following, the facial structure of M^* is described as the facial structure of the oriented matroid $\mathbb{A}\mathrm{ff}\,(\tilde{E})$, where $\tilde{E}$ is a finite set of points. Since only the positive circuits of M are relevant, several non-isomorphic oriented matroids will, in general, be attached to each example. This means that, in most of the figures, certain points of $\tilde{E}$ may have additional dependences without modifying the facial structure of M^* (leading possibly to a non-realizable situation). For instance, the central point of Figure 1(a) may be aligned with two other points of the configuration. In the particular case where no additional dependence can occur, i.e., when all of the signed circuits of M^* can be recovered from its facial structure, the (totally cyclic) oriented matroid M^* is said to be *rigid* (see [5, Section 9]). Lemmas 4.5–4.8 below, which will be used in the proofs of Reay's conjecture in $\mathbb{R}^4$ and $\mathbb{R}^5$, characterize d-lopsided oriented matroids of rank $d = 4$ or 5 such that $|E| = d+3$ or $d+4$. Similar characterizations for $d = 2$ and 3 are left to the reader as an easy exercise.

LEMMA 4.5. *Let M be a totally cyclic rank* 4 *oriented matroid on a* 7*-element set E. Then M is* 4*-lopsided if and only if M^* has the facial structure of one of the sets of $\mathbb{R}^2$ given in Figure* 1.

PROOF. We recall that every rank 3 oriented matroid on E with $|E| \leq 8$ is realizable [5, Section 6]. Thus, M^* can be represented as the oriented matroid of affine dependences of

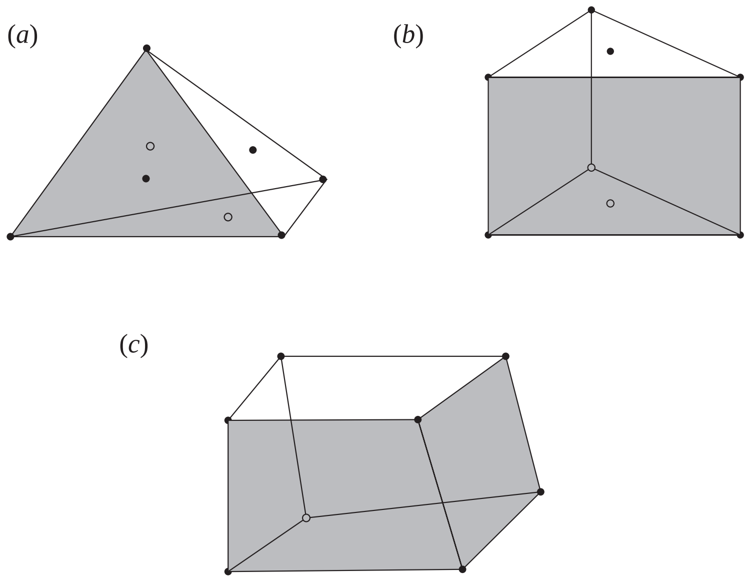

FIGURE 2.

a 7-element set $\tilde{E}$ of (possibly multiple) points of $\mathbb{R}^2$. With the notational convention given in Section 2, the lopsidedness condition amounts to the following:

— C is a positive circuit of M if and only if there is an edge $\tilde{F}$ of conv $\tilde{E}$ such that $\tilde{C} = \tilde{E} \backslash \tilde{F}$; in particular, each edge of conv $\tilde{E}$ contains at least three points of $\tilde{E}$;
— the intersection of any two edges of conv $\tilde{E}$ cannot contain two points of $\tilde{E}$: this means that the vertices of conv $\tilde{E}$ cannot be multiple points of $\tilde{E}$.

Thus, conv $\tilde{E}$ is a triangle, which leads to one of the two situations described in Figure 1, hence to the two possible cases of the lemma. □

LEMMA 4.6. *Let M be a totally cyclic rank* 4 *oriented matroid on an* 8*-element set E. Then M is* 4*-lopsided if and only if M^* has the facial structure of one of the sets of* $\mathbb{R}^3$ *given in Figure* 2.

PROOF. Suppose first that M is realizable. Then, M^* can be represented as the oriented matroid of affine dependences of an 8-element set $\tilde{E}$ of (possibly multiple) points of $\mathbb{R}^3$. The hypotheses translate as follows:

— C is a positive circuit of M if and only if there is a 2-face $\tilde{F}$ of conv $\tilde{E}$ such that $\tilde{C} = \tilde{E} \backslash \tilde{F}$; in particular, each facet of conv $\tilde{E}$ contains at least four points of $\tilde{E}$;
— the intersection of any two facets of conv $\tilde{E}$ (and in particular each edge of conv $\tilde{E}$) cannot contain three points of $\tilde{E}$.

By listing all 3-polytopes with up to eight vertices, this situation appears in exactly three cases, depicted in Figure 2.

In the general case, although M^* may not be realizable, its face lattice can be represented as the face lattice of a 3-polytope, by Steinitz's theorem. It follows that the facial structure of M^* is still given by one of the drawings of Figure 2. Note that, by the results of Bokowski and Richter (see [6]), non-realizability actually occurs in exactly 24 cases. □

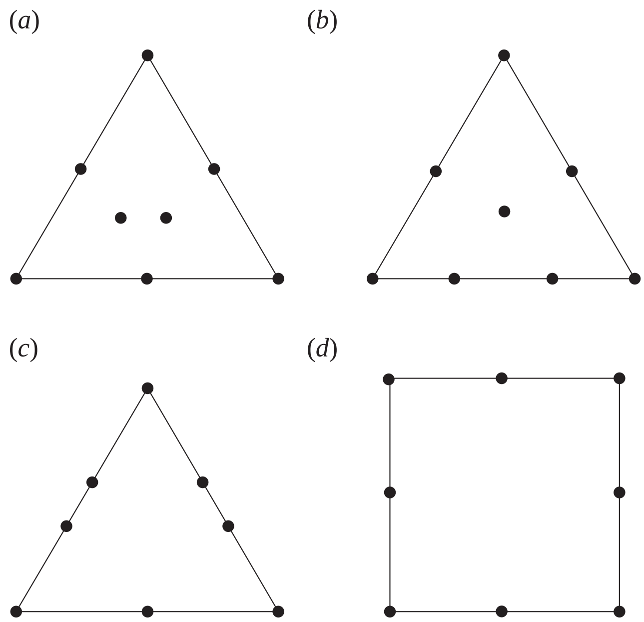

FIGURE 3.

Similarly we obtain the following lemmas.

LEMMA 4.7. *let M be a totally cyclic rank* 5 *oriented matroid on an* 8*-element set E. Then M is* 5*-lopsided if and only if M^* has the facial structure of one of the sets of $\mathbb{R}^2$ given in Figure* 3*.*

LEMMA 4.8. *Let M be a totally cyclic rank* 5 *oriented matroid on a* 9*-element set E. Then M is* 5*-lopsided if and only if M^* has the facial structure of one of the sets of $\mathbb{R}^3$ obtained from Figure* 2 *by adding a point which is either interior to* $\operatorname{conv} \tilde{E}$ *or in the relative interior of one of its faces.*

EXAMPLE 4.9. Let S be a simplex of $\mathbb{R}^{d-1}$, identified with its set of vertices $\{a_1, a_2, \ldots, a_d\}$. For each i, we consider a point a'_i, interior to $\operatorname{conv} S$ and close to the facet $\operatorname{conv}(S \backslash a_i)$. We also set $o = \frac{1}{d} \cdot \sum_{k=1}^{d} a_k$ and let ∞ be a point at infinity. Let $\mathcal{C}$ be the d-dimensional cell complex whose d-faces are precisely $\operatorname{conv}(S \backslash a_i) \cup \{a'_i, o\}$, $1 \leq i \leq d$ and $(S \backslash \{a_i, a_j\}) \cup \{a'_i, a'_j, \infty\}$, $1 \leq i < j \leq d$ (see Figure 4). Then, $\mathcal{C}$ is the Schlegel diagram of a certain d-polytope P with $2d + 2$ vertices. Moreover, each facet (resp. $(d-1)$-face) of P has exactly $d + 1$ (resp. $d - 1$) vertices. Defining M^* to be the oriented matroid of affine dependences of the vertices of P, we get by Lemma 4.4 that M is a totally cyclic $(d + 1)$-lopsided oriented matroid of rank $d + 1$. Finally, M satisfies assertion (i) of the lopsidedness conjecture since $\{o, \infty\}$ is clearly a half-space of M.

EXAMPLE 4.10. A third example of a 5-lopsided oriented matroid of rank 5 on 10 elements can also be described by a Schlegel diagram. We suspect that, up to the above-mentioned considerations of rigidity, no other extremal example can be found for the parameters $k = d = 5$. Consider a regular octahedron of centre o, and select two opposite faces, the vertices

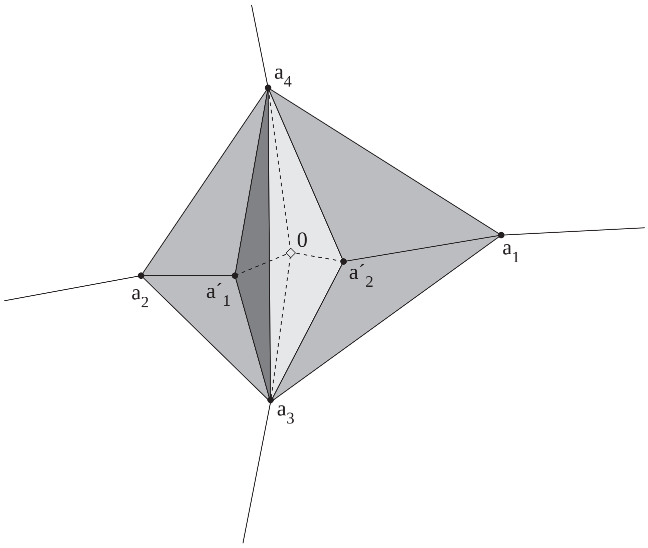

FIGURE 4.

of which are denoted a_1, a_2, a_3 and b_1, b_2, b_3, respectively. Let a_4 and b_4 be points of $\mathbb{R}^3$, placed 'above' those faces. Finally, let again ∞ denote a point at infinity. We define the cell complex $\mathcal{C}$ by the list of its three-dimensional faces:

$$oa_1a_2a_3a_4, \quad ob_1b_2b_3b_4, \quad oa_1a_2b_1b_2, \quad oa_3a_3b_2b_3, \quad oa_3a_1b_3b_1,$$
$$\infty a_1b_1b_2b_4, \quad \infty a_2b_2b_3b_4, \quad \infty a_3b_3b_1b_4,$$
$$\infty a_1a_2a_4b_2, \quad \infty a_2a_3a_4b_3, \quad \text{and} \quad \infty a_3a_1a_4b_1.$$

Then, $\mathcal{C}$ is the Schlegel diagram of a 4-polytope with vertex set $\tilde{E}$ and whose facets are isomorphic to a bipyramid. Defining M^* as $\mathbb{A}\mathrm{ff}(\tilde{E})$ yields a 5-lopsided oriented matroid of rank 5 on 10 elements having $\{o, \infty\}$ as a half-space.

5. SPECIAL CASES OF THE LOPSIDEDNESS CONJECTURE

We now prove Conjecture 4.1 for the values $k = 1, 2, 3, d + 1$ and $(k, d) = (4, 4)$. For the sake of convenience, the case $k = d + 1$ is treated separately.

THEOREM 5.1. *The lopsidedness conjecture is true for $k = d + 1$ and arbitrary d. Moreover, any totally cyclic $(d + 1)$-lopsided oriented matroid of rank d is isomorphic to $M_{d+1,d}$, i.e., is reduced to a single positive circuit.*

PROOF. When $k = d + 1$, the condition on the cardinality of positive circuits is irrelevant in the definition of k-lopsidedness. As k-positive independence implies $(k - 1)$-positive independence, it follows that $M(A)$ is d-lopsided for each $A \subseteq E$ of rank $\geq d - 1$.

Suppose first that M is not totally cyclic. Then, M has a positive cocircuit D and we define H to be the hyperplane $E \backslash D$. Applying an induction hypothesis to H yields a half-space H^- of $M(H)$ such that $|H^-| \leq 1$, and H^- is also a half-space of M since no element of D belongs to a positive circuit.

If M is totally cyclic, let $B = B_1 \cup B_2 \cup \cdots \cup B_p$ a positive base of M, decomposed as in Lemma 3.1. If we had $p \geq 2$, then $|B_1 \cup B_2| \leq d + 2$, hence B_1 and $B'_1 \cup B_2$ would be positive circuits of M for some $B_1' \subseteq B_1$: impossible. Thus, $p = 1$ and $|B| = |B_1| = d+1$. If $B \neq E$, then for each $e \in E\backslash B$, there would exist a positive circuit C such that $e \in C \subseteq B \cup e$, hence $|B \cup C| \leq d+2$, contradicting the $(d+1)$-positive independence of M. It follows that $E = B$ and a half-space of M is obtained by taking $\{x\}$ with $x \in E$. □

THEOREM 5.2. *The lopsidedness conjecture is true in the following cases:*

— $k = 1, 2, 3$ *and arbitrary* d*;*
— $k = d = 4$.

PROOF. The result is clear for $k = 1$ since the conditions of Conjecture 4.1 imply in this case that M consists of d independent elements together with at most one loop.

We proceed by induction on d, $k \geq 2$ being fixed. When $d = k - 1$, the result follows from Theorem 5.1. We begin by general remarks which hold for every value of k and might be helpful for a general proof of the conjecture.

Suppose first that M is not totally cyclic. Then, M has a positive cocircuit D and, as in the proof of the case $k = d + 1$, applying the induction hypothesis to the hyperplane $H := E\backslash D$ yields a half-space H^- of $M(H)$, hence of M, such that $|H^-| \leq d + 1 - k$.

If M has a loop x, then M has no other positive circuit by k-positive independence, hence $\{x\}$ is a half-space of M. Furthermore, if M is cyclic, then clearly $|E\backslash x| \leq d$, so that $|E| \leq d + \binom{d}{k-1}$.

How about: if M is not connected, say the direct sum of $M_1 = M(E_1)$ and $M_2 = M(E_2)$ of ranks $d_1 \geq k-1$ and $d_2 \geq k-1$, respectively, the induction hypothesis yields half-spaces E_1^- and E_2^- of M_1 and M_2 such that $|E_1^-| \leq d_1+2-k$ and $|E_2^-| \leq d_2+2-k$. Then, $E_1^- \cup E_2^-$ is a half-space of M such that $|E_1^- \cup E_2^-| \leq d_1+2-k+d_2+2-k \leq d+2-k$. Moreover, if M is cyclic, so are M_1 and M_2, hence $|E| \leq |E_1|+|E_2| \leq d_1 + \binom{d_1}{k-1} + d_2 + \binom{d_2}{k-1} \leq d + \binom{d}{k-1}$. If $d_1 < k - 1$, say, we can replace $d_1 + 2 - k$ by 1 and use similar calculations.

In the following, we assume that M is totally cyclic and connected and that $2 \leq k < d+1$. In particular, M is not reduced to a single circuit. We show that we can conclude if M satisfies the following unicity property: (5.1.1) every positive circuit C has an element x such that C is the only positive circuit containing x.

Let C be such a positive circuit and $y \in C\backslash x$. Then, $M\backslash y$ is not cyclic, so there is a hyperplane H_y of M such that $D_y^- = \{y\}$ and $x \in D_y^+$, where D_y denotes the cocircuit $E\backslash H_y$. By the induction hypothesis, we can find a half-space H_y^- of H_y such that $|H_y^-| \leq d-k+1$, hence $H_y^- \cup y$ is a half-space of M satisfying $|H_y^- \cup y| \leq d - k + 2$, which proves (i).

Let A be the set of elements of E which belong to at least two positive circuits. We first show that A is independent. Condition (5.1.1) and the connectedness of M imply, via Lemma 3.2 (applied to $E_1 = C$), that $C \cap A \neq \phi$ and $C\backslash A \neq \phi$ for every positive circuit C. If $M(A)$ contains some signed circuit C_1, then C_1 cannot be positive. Starting from C_1 and performing successive sign eliminations with positive circuits of M produces a sequence $C_1, C_2, \ldots, C_p$ of circuits satisfying $\phi = C_p^- \subset C_{p-1}^- \subset \cdots \subset C_1^-$. We observe that the last operation consists of eliminating an element $a \in C_{p-1}^- \subseteq A$ by using a positive circuit C containing a. Let $x \in C\backslash A$, then $x \notin C_{p-1}^-$, so we can choose C_p such that $x \in C_p$, which is absurd since x belongs to only one positive circuit. We deduce from the preceding observations that M/A is a totally cyclic oriented matroid of rank $d - |A|$ and that every element e of $E\backslash A$ belongs to a positive circuit of cardinality $\leq k - 1$. Conversely, for every positive circuit C' of M/A with $e \in C'$, there is a signed circuit C'' of M such that $C' = C''\backslash A$, which must also be positive, by orthogonality with the cocircuits $D_y,\ y \in C'' \cap A$. It follows that C' and C'' are unique with respect to these conditions. As a consequence, the positive circuits of M/A form a partition of $E\backslash A$. Let L denote the set of loops of M/A. If C_1'' and C_2'' are positive circuits of M such that $C_1''\backslash A$ and $C_2''\backslash A$ are loops of M/A, then $|(C_1'' \cup C_2'')\backslash A| \leq k-1$ by k-positive independence, hence $|L| \leq \binom{|A|}{k-1}$. Finally, as positive circuits of cardinality $\geq$2 form a direct sum by Lemma 3.2, we find that $|E\backslash(A \cup L)| \leq 2 \cdot \operatorname{rank}(M/A) = 2 \cdot (d - |A|)$. Then,

$$|E| = |A| + |L| + |E\backslash(A \cup L)| \leq |A| + \binom{|A|}{k-1} + 2 \cdot (d - |A|)$$

and an easy calculation shows that this quantity is bounded above by $d + \binom{d}{k-1}$ with equality if and only if $|A| = d$ and $|L| = \binom{d}{k-1}$. This proves (ii) when M satisfies (5.1.1). Note that, under this hypothesis, equality in (ii) holds if and only if M and $M_{k.d}$ have the same facial structure.

We now complete the proof by separating the different remaining cases.

Case 1. $k = 2$.

The presence of two positive circuits of cardinality $\leq$2 containing a given element of E is incompatible with the 2-positive independence of M. It follows from Lemma 3.2 that, except when $d = 2$, M cannot be connected. More precisely, the first part of the proof shows that M is the direct sum of 2-element positive circuits and coloops. In the particular case where M is totally cyclic, this result implies the well-known fact that a positive base B of $\mathbb{R}^d$ has $2d$ elements if and only if B is $\{\lambda_1\vec{e}_1, \mu_1\vec{e}_1, \lambda_2\vec{e}_2, \mu_2\vec{e}_2, \ldots, \lambda_d\vec{e}_d, \mu_d\vec{e}_d\}$, where $(\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_d)$ is a basis of $\mathbb{R}^d$ and $\lambda_i\mu_i < 0$ for all i (see [2, 7, 9]).

Case 2. $k = 3$.

We assume $d \geq 3$ and the existence of a positive circuit C_0 of M such that each element of C_0 is contained in another positive circuit of M. Necessarily, C_0 has cardinality 3. Setting $C_0 = xyz$, there exist a, b, c, d, e, f in E such that xab, ycd and zef are positive circuits distinct from C_0. These six elements may not be distinct but we have $\{x, y, z\} \cap \{a, b, c, d, e, f\} = \phi$ by 3-positive independence. Applying sign elimination between $\overline{xyz}$ and xab, we first obtain a circuit C such that $a \in C^+ \subseteq \{a, b\}$ and $C^- \subseteq \{y, z\}$; then, using ycd and zef if necessary, we obtain a positive circuit C_1 such that $a \in C_1 \subseteq \{a, b, c, d, e, f\}$. By 3-positive independence, $|C_1 \cap \{a, b\}| = |C_1 \cap \{c, d\}| = |C_1 \cap \{e, f\}| = 1$ and we fix the notation such that $C_1 = ace$. If a, b, c, d, e, f were not distinct with, for instance, $b = d$, we would have $|C_1 \cup C_2| \leq 4$ by similarly introducing a positive circuit C_2 such that $b \in C_2 \subseteq A$: impossible. Let us keep the same definition for C_2 in the following. If we had $|C_1 \cap C_2| = 1$, say $C_1 \cap C_2 = \{e\}$, then we could restart the procedure with $(a, c, e, b, x, d, y, b, c)$ instead of $(x, y, z, a, b, c, d, e, f)$, which leads to a contradiction since b appears twice in the new list. It follows that $C_1 \cap C_2 = \phi$, hence $C_2 = bdf$. Now, set $A = \{x, y, z, a, b, c, d, e, f\}$. By 3-positive independence, the only positive circuits of $M(A)$ are those which have already been determined, i.e., C_0, C_1, C_2, xab, ycd and zef. Moreover, we note that the elements of A play symmetric roles with respect to those positive circuits.

Suppose that $E \neq A$. By Lemma 3.2, there exists a positive circuit C such that $C \cap A \neq \phi$ and $C\backslash A \neq \phi$. The situation reduces to $|C \cap A| = 2$ for if $|C \cap A| = 1$ with $C \cap A = \{x, a', b'\}$ for instance, we can apply the preceding arguments to $(x, y, z, a', b', c, d, e, f)$ and find a positive circuit C' such that $C' \cap \{a', b'\} = \{a'\}$ hence $|C' \cap A| = 2$. Using 3-positive independence and the symmetries of $M(A)$, we may assume that C has the form azf' with $f' \notin A$. Restarting the procedure with $(x, y, z, a, b, c, d, a, f')$ then leads to a contradiction since a appears twice in this list.

To conclude, we just have to verify that M satisfies condition (i). We observe that M^* is acyclic and that every element of E belongs to exactly four facets of M^*. This implies rank $M^* \leq 5$, hence rank $M \geq 4$. Finally, $E^- = \{x, c, f\}$ is a half-space of M such that $|E^-| \leq d - 1$.

Case 3. $k = d = 4$.

Suppose first that M has two different positive circuits C_1 and C_2 of cardinality 3 or less. By 4-positive independence, we have $|C_1| = |C_2| = 3$, $C_1 \cap C_2 = \phi$ and $C_1 \cup C_2$ contains no other positive circuit. The discussion of case 2 shows that $M(C_1 \cup C_2)$ cannot have rank 3 or less, and it follows easily that $M(C_1 \cup C_2)$ is the direct sum of $M(C_1)$ and $M(C_2)$. Then, $C_1 \cup C_2$ is a positive base of M and the presence of an element x in $E \backslash (C_1 \cup C_2)$ leads to the existence of a positive circuit C_x such that $x \in C_x \subseteq C_1 \cup C_2 \cup x$, hence $|C_1 \cup C_x| \leq 5$ or $|C_1 \cup C_x| \leq 5$: a contradiction.

Assume now that M has exactly one positive circuit of cardinality ≤ 3, that we denote by C_2. As M is not the direct sum of $M(C_2)$ and $M(E \backslash C_2)$, there exists a positive circuit $C_1 \neq C_2$ such that $C_1 \cap C_2 \neq \phi$. Then, $|C_1| = 4$, $|C_2| = 3$ and $|C_1 \cap C_2| = 1$ by 4-positive independence. We set $C_2 = \{a, b, c\}$ with $a \in C_1 \cap C_2$. Moreover, $C_1 \cup C_2$ is a positive base of M (note that we can take $B_1 = C_1$ and $B_2 = C_2 \backslash C_1$ with the notation of Lemma 3.1). For any $x \in E \backslash (C_1 \cup C_2)$, we denote by C_x the (unique) positive circuit such that $x \in C_x \subseteq C_1 \cup C_2 \cup x$. Applying Lemma 4.5 to $M(C_1 \cup C_2 \cup x)$ yields $|C_1 \cap C_x| = 2$, $|C_2 \cap C_x| = 1$ and $C_1 \cap C_2 \cap C_x = \phi$ (the situation of Figure 1(b) is obtained). If there exist two elements x, y of $E \backslash (C_1 \cup C_2)$ such that C_x and C_y contain the same element of C_2, say b, then $|C_x \cap C_y| = 2$ (by Lemma 4.5 again), hence $C_x \cup C_y$ is also a positive base of M. In particular, there is a positive circuit C such that $c \in C \subseteq C_x \cup C_y \cup c$. Since all the positive circuits of $M(C_1 \cup C_2 \cup x)$ and $M(C_1 \cup C_2 \cup y)$ are known, C necessarily contains x, y and a fourth element of C_1, hence $|C_1 \cap C| = 1$: a contradiction. It follows that $|E \backslash (C_1 \cup C_2)| \leq 2$, i.e., $|E| \leq 8$. The case $|E| \leq 7$ is immediate by Lemma 4.5. If $|E| = 8$, we set $E \backslash (C_1 \cup C_2) = \{x, y\}$. The circuits C_x and C_y can be written $xbde$ and $ycde$, respectively, with d, e in C_1 (use again Lemma 4.5). The only other possible positive circuit of M is then $xyaf$, where f is the fourth element of C_1 and $\{a, e\}$ is a half-space of M.

Finally, suppose that every positive circuit of M has exactly four elements. By Lemma 4.5, the intersection of any two positive circuits of M has an even cardinality. In particular, any positive base of M is of the form $C_1 \cup C_2$, where C_1 and C_2 are positive circuits such that $|C_1 \cap C_2| = 2$. Denoting again by C_x the unique positive circuit such that $x \in C_x \subseteq C_1 \cup C_2 \cup x$, we have $|C_1 \cap C_x| = |C_2 \cap C_x| = 2$, hence there is precisely one element of C_x in each of the sets $C_1 \backslash C_2$, $C_1 \cap C_2$ and $C_2 \backslash C_1$. An easy discussion shows that, if $E \backslash (C_1 \cup C_2)$ has three distinct elements x, y, z, then $|C_x \cap C_y|$, $|C_y \cap C_z|$ and $|C_z \cap C_x|$ could not all be even, what we have seen to be impossible. It follows that $|E \backslash (C_1 \cup C_2)| \leq 2$, i.e., $|E| \leq 8$. The case $|E| \leq 7$ is again immediate. Finally, if $|E| = 8$, we set $E \backslash (C_1 \cup C_2) = \{x, y\}$. As the only other positive circuits of M that have not yet been mentioned must contain both x and y, we deduce that $E^- = \{a, y\}$ is a half-space of M, where a is the unique element of $C_1 \cap C_2 \cap C_x$. □

REMARK 5.3. For $k = 1, 2, 3, d + 1$, the above proof shows that the oriented matroids on $d + \binom{d}{k-1}$ elements satisfying the hypotheses of the lopsidedness conjecture have the facial structure of $M_{k,d}$. The rigidity of $M_{k,d}$ can easily be established for $k = 1, 2, d + 1$ and $(k, d) = (3, 3)$, ensuring uniqueness (up to isomorphism) of the extremal examples in each of these cases. On the other hand, Lemma 4.6 shows that three different facial structures may be found for $k = d = 4$. Moreover, their corresponding oriented matroids are not rigid. We note that $(M_{4,4})^*$ is the oriented matroid of affine dependences of the set given by Figure 2(a), in which the tetrahedron is regular and the four remaining points are on the centre of its faces.

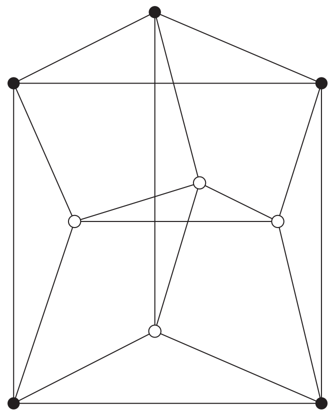

FIGURE 5.

REMARK 5.4. The oriented matroid M obtained during the discussion of Case 2 is not unique *a priori* since, again, we have only determined its positive circuits. More precisely, M^* has the facial structure of the 4-polytope with nine vertices and six facets (all of them being 3-prisms), depicted in Figure 5 (which proves, in passing, the existence of M). In fact, it turns out that M^* is rigid and, in particular, M is unique, up to isomorphism. Furthermore, M is isomorphic to the cographic oriented matroid $\mathbb{B}(K_{3,3})$ (see Example 4.3). We sketch the proof of this result. First, M^* has six cocircuits of cardinality 3, corresponding to the complements of its facets, i.e., to the triangular faces of Figure 5. M^* also has nine circuits of cardinality 4, each associated with a 4-sided face. By orthogonality with the 3-element cocircuits, it is easily checked that M^* has no other circuit of cardinality 5 or less, and has at most six circuits of cardinality 6. On the other hand, using the elimination axiom between 4-element circuits, we find that M^* has at least six circuits of cardinality 6. At this point, we have determined the underlying matroid of M^* and shown that it is isomorphic to the graphic matroid of $K_{3,3}$. Finally, the (signed) orthogonality property with the positive cocircuits of cardinality 3 forces the orientation of $K_{3,3}$ to be that chosen in Example 4.3, hence M is isomorphic to $\mathbb{B}(K_{3,3})$.

6. (m, k)-DIVISIBILITY

The end of the paper is devoted to divisibility properties for sets of points with independence properties. Concerning k-independence, Theorem 6.1 below settles Conjecture 1.4 for $k \leq 3$ in a slightly sharper form.

THEOREM 6.1. *Let m, j and k be integers such that $m \geq 2$ and $0 \leq j \leq k \leq 3$, let X be a set of $(m-1)\cdot(2d-k+1)+j+1$ points in $\mathbb{R}^d$ and let ω be a Tverberg point of X such that X is k-positively independent at ω. Then, X is (m, j)-divisible at ω.*

PROOF. We proceed by (finite) induction on m and j, the result following from Tverberg's theorem for $j = 0$. By the induction hypothesis, there exist pairwise disjoint subsets $S_1, S_2, \ldots, S_m$ of X with the following properties:

(i) $N := \cap_{i=1}^{m} \operatorname{conv} S_i$ is at least $(j-1)$-dimensional and contains ω;
(ii) $S_1, S_2, \ldots, S_m$ are inclusion-minimal with respect to condition (i);
(iii) $(|S_1|, |S_2|, \ldots, |S_m|)$ is lexicographically maximal with respect to conditions (i) and (ii).

In the proof, we may assume that N is exactly $(j-1)$-dimensional and that ω is in general position in the relative interior of N by the perturbation principle (Lemma 3.9). The result is also immediate if $|S_1| = d+1$ (in this case, $\omega \in \operatorname{int} S_1$ and the induction hypothesis applies to $X \setminus S_1$), so we may also assume $|S_1| \leq d$ in what follows. Finally, we note that Lemma 3.4 implies that $X \cap \operatorname{aff} N = \phi$.

Case 1. $|S_m| \geq k+1$.

As $X \cap \operatorname{aff} N = \phi$, we may consider a hyperplane H of $\mathbb{R}^d$ containing $\operatorname{aff} N$ but no point of X. Then, H separates $Y := X \setminus \bigcup_{i=1}^m S_i$ into two parts, one of them, say Y', being of cardinality at least $\frac{1}{2} \cdot ((m-1) \cdot (2d-k+1) + j + 1 - \sum_{i=1}^m |S_i|)$, hence:

$$\left| Y' \cup \bigcup_{i=1}^{m} S_i \right| \geq \frac{1}{2} \cdot \left((m-1) \cdot (2d-k+1) + j + 1 + \sum_{i=1}^{m} |S_i| \right).$$

Since $|S_i| \leq k+1$ for all $i \leq m$ by condition (iii), we obtain

$$\left| Y' \cup \bigcup_{i=1}^{m} S_i \right| \geq \tfrac{1}{2} \cdot ((m-1) \cdot (2d-k+1) + j + 1 + m \cdot (k+1)) \geq (m-1) \cdot (d+1) + j + 1$$

and Lemma 3.6 applies to $X' := Y' \cup \bigcup_{i=1}^m S_i$. Note that condition (v) of Lemma 3.6 is void and that [13, Lemma 3.1] can be used in a simpler way here.

Case 2. $|S_m| \leq k$ and $j = 1$.

The lexicographical choice of $(|S_1|, |S_2|, \ldots, |S_m|)$ implies that every inclusion-minimal subset S of $E := X \setminus \bigcup_{i=1}^{m-1} S_i$ such that $\omega \in \operatorname{conv} S$ satisfies $|S| \leq |S_m| \leq k$. The oriented matroid $M := \mathbb{A}\mathrm{ff}(E \cup \omega)/\omega$ is then k-lopsided of rank at most d. By Theorem 5.2, M has a half-space E^- such that $|E^-| \leq d+2-k$. This means that there is a hyperplane H passing through ω such that $|H^- \cap E| \leq d+2-k$, where H^- denotes one of the two closed half-spaces defined by H. Then $X' := E \setminus H^-$ and ω are strictly separated by H and, since the other half-space contains at least one element of S_m, we can write:

$$\left| X' \cup \bigcup_{i=1}^{m} S_i \right| \geq (m-1)\cdot(2d-k+1)+j+1-(d-k+2)+1 = (m-1)\cdot(d+1)+(m-2)\cdot(d-k)+j.$$

2.1. If $m \geq 3$ and $k < d$, Lemma 3.6 directly applies to $X' \cup \bigcup_{i=1}^m S_i$.

2.2. If $k = d$, we show that ω and Y are weakly separated by a hyperplane. If this was not the case, then M would be totally cyclic. We note that $|S_m| = k$ for, otherwise, M would have another positive circuit S, hence $|S \cup S_m| \leq 2k-2 \leq k+1$, which is impossible by k-positive independence. On the other hand, as $\omega \in \operatorname{int} \operatorname{conv} Y$, we also find that $|Y| \geq d+1$, hence $|E| \geq 2d+1$, contradicting the bound $|E| \leq 2d$ obtained in Theorem 5.2. It follows that ω and Y are weakly separated by a hyperplane and we can conclude by Lemma 3.7.

2.3. Finally, if $m = 2$ and if X was not $(2,1)$-divisible at ω, the same would hold for $X \cup \omega$. As $|X \cup \omega| = 2d-k+4 \geq 2d+1$, $X \cup \omega$ would be a Reay set by [13, Theorem 4.1], which means that X is the $(2d)$-element set obtained by taking a point on each half-line defined by d affinely independent lines passing through ω. Moreover, the equality $|X| = 2d$ requires $k = 3$. But the 3-positive independence of X is then contradicted since all positive circuits of $\mathbb{A}\mathrm{ff}(X \cup \omega)/\omega$ have exactly two elements.

Case 3. $|S_m| \leq k$ and $j \geq 2$.

By Lemma 3.4, this situation can only occur when $j = 2$ and $|S_m| = k = 3$. If aff N and Y are strictly separated by a hyperplane, then Lemma 3.6 directly applies to X. In the contrary case, let Y' be a subset of Y such that $\operatorname{aff} N \cap \operatorname{conv} Y' \neq \phi$ and is inclusion-minimal with respect to this condition. Then, $\omega \in \operatorname{relint} \operatorname{conv} Y' \cup S_m$, so there is a subset $S \neq S_m$ of $Y' \cup S_m$ such that $\omega \in \operatorname{relint} \operatorname{conv} S$. The lexicographical choice of $(|S_1|, |S_2|, \ldots, |S_m|)$ implies $|S| \leq 3$. Moreover, by the perturbation principle, $\operatorname{conv} S \cap \operatorname{conv} S_m$ is at least one-dimensional. It follows from Lemma 3.4 again that $S \cap S_m = \phi$ and that aff S and aff S_m are two distinct planes such that $\operatorname{aff} S \cap \operatorname{aff} S_m = \operatorname{aff} N$. The oriented matroid $M := \mathbb{A}\mathrm{ff}(S \cup S_m \cup \omega)/\omega$ is then totally cyclic, 3-lopsided of rank 3. As $|S \cup S_m| = 6$, M is thus isomorphic to $M_{3,3}$, which is impossible since $M_{3,3}$ does not have disjoint positive circuits. □

REMARK 6.2. In the preceding reasoning, the condition $k \leq 3$ has only been used in cases 2.2 and 3. It follows that, if assertion (i) of the lopsidedness conjecture is true for parameters k and d such that $k < d$, then every set X of $(m-1) \cdot (2d - k + 1) + 2$ points in $\mathbb{R}^d$, such that X is k-positively independent at some Tverberg's point ω, is $(m-1)$-divisible at ω. More generally, we conjecture that the condition $k \leq 3$ can be dropped in the hypotheses of Theorem 6.1, i.e., every set of $(m-1) \cdot (2d - k + 1) + j + 1$ points in $\mathbb{R}^d$ $(0 \leq k \leq d)$, such that X is k-positively independent at one of its Tverberg's points ω, is (m, j)-divisible at ω.

7. PROOF OF REAY'S CONJECTURE IN DIMENSION 4

In [12], we have given a purely geometric proof of Reay's conjecture in three-dimensional space. We now apply the preceding results to settle the case $d = 4$. We point out that a similar treatment would also work for $\mathbb{R}^3$, leading to a shorter proof than that given in [12] (note that Theorem 6.1 also supplies a proof of the case $d = k = 3$). Again, we shall prove a more precise theorem, analogous to the results obtained in smaller dimensions (see [4, 12]).

THEOREM 7.1. *Any set X of $5 \cdot (m-1) + k + 1$ points in general position in $\mathbb{R}^4$, with $0 \leq k \leq 4$, is (m, k)-divisible. More precisely, if ω is a Tverberg point of X, and if X is only assumed to be 4-positively independent at ω, then X is (m, k)-divisible at ω.*

PROOF. We proceed by induction on k and m. The result holds for evident reasons for $m = 1$, and the case $k = 0$ follows from Tverberg's theorem, so we consider in the proof a subset X of $5 \cdot (m-1) + k + 1$ points in $\mathbb{R}^4$, 4-positively independent at ω, with $m \geq 2$ and $1 \leq k \leq 4$. By the induction hypothesis, let $S_1, S_2, \ldots, S_m$ be pairwise disjoint subsets of X such that $N := \bigcap_{i=1}^m \operatorname{conv} S_i$ is at least $(k-1)$-dimensional, contains ω, and is inclusion-minimal with respect to these properties. By the perturbation principle (Lemma 3.9), we may assume ω to be in general position in the relative interior of N. As in the proof of Theorem 6.1, we again make the technical assumption that $(|S_1|, |S_2|, \ldots, |S_m|)$ is lexicographically maximal with respect to these conditions. In particular, we have $5 \geq |S_1| \geq |S_2| \geq \cdots \geq |S_m|$. There are three cases in which the conclusion is immediate:

- if N is already at least k-dimensional;
- if $|S_1| = 5$ (then, $\omega \in \operatorname{int} S_1$ and we apply the induction hypothesis to $X \setminus S_1$);
- if aff N and $Y := X \setminus \bigcup_{i=1}^m S_i$ are weakly separated by a hyperplane (Lemma 3.7 applies).

In the following, we suppose that none of these three cases apply and we show that a contradiction can be obtained.

Case 1. $k = 1$.

$S_1, S_2, \dots, S_m$ being fixed, the preceding assumptions imply that N is reduced to ω, $|S_i| \leq 4$ for all i and $\omega \in \operatorname{conv} Y$. The last condition means that there exists a minimal $(m+1)$th simplex $S_{m+1} \subseteq Y$ such that $\omega \in \operatorname{conv} S_{m+1}$. The lexicographical choice of $(|S_1|, |S_2|, \dots, |S_m|)$ already shows that $|S_{m+1}| \leq 4$. Similarly, any minimal subset S of $E := X \setminus \bigcup_{i=1}^{m-1} S_i$ such that $\omega \in \operatorname{conv} S$ satisfies $|S| \leq 4$. The oriented matroid $M := \mathbb{A}\mathrm{ff}(E \cup \omega)/\omega$ is then 4-lopsided of rank (at most) 4. By Theorem 5.2, we deduce that $|E| \leq 8$. On the other hand, Lemma 4.5 shows that $|E| = 7$ is impossible since S_m and S_{m+1} are disjoint positive circuits of M. In view of 4-positive independence, only two situations remain:

— $E = S_m \cup S_{m+1}$ and $|S_m| = |S_{m+1}| = 3$; or
— $E = S_m \cup S_{m+1}$ and $|S_m| = |S_{m+1}| = 4$;

and, in each case, Lemma 3.7 applies: impossible.

Case 2. $k = 2$.

By 4-positive independence and one-dimensional intersection, we have $|S_i| + |S_j| \geq 7$ for all $i \neq j$, hence $|S_i| = 4$ for $i \leq m-1$ and $|S_m| = 3$ or 4. Since Y and $\operatorname{aff} N$ are not weakly separated by a hyperplane, we note that $|Y| \geq 4$. Now, setting $E := S_1 \cup Y$, the oriented matroid $M := \mathbb{A}\mathrm{ff}(S_1 \cup Y \cup \omega)/\omega$ is then totally cyclic, 4-lopsided of rank 4. It follows from Theorem 5.2 that $|E| \leq 8$, hence $|Y| = 4$. By Lemma 4.6, M^* is then isomorphic to the oriented matroid of affine dependences of a subset $\tilde{E}$ of $\mathbb{R}^3$ given by one of the drawings of Figure 2. Using the notational conventions of Section 2, S_1 is a positive circuit of M amounts to saying that $\operatorname{conv} \tilde{Y}$ is a face of $\operatorname{conv} \tilde{S}_1 \cup \tilde{Y}$; let $\tilde{F}$ and $\tilde{F}'$ be the two other faces containing a given vertex of $\operatorname{conv} \tilde{Y}$. Then, $S := E \setminus F$ and $S' := E \setminus F'$ are two positive circuits of M such that $|S| = |S'| = 4$ and $S \cap S' = \{x, y\}$, with $x \in S_1$ and $y \in Y$. The hyperplanes $\operatorname{aff} S$ and $\operatorname{aff} S'$ of $\mathbb{R}^4$ contain x, y and ω, hence x, y and $\operatorname{aff} N$ (by the perturbation principle), and are distinct by Lemma 3.4. Their intersection is thus a plane P containing x, y and $\operatorname{aff} N$. Since $x \notin \operatorname{aff} N$ by Lemma 3.4 again, we find that $P = \operatorname{aff}(N \cup x)$. It follows that $y \in P$, hence $y \in \operatorname{aff} S_1$: impossible.

Case 3. $k = 3$.

4-positive independence and two-dimensional intersection now imply $|S_i| + |S_j| \geq 8$ for all $i \neq j$, hence $|S_i| = 4$ for all i. Since Y and $\operatorname{aff} N$ are not weakly separated by a hyperplane, we have $|Y| \geq 3$. Let Z be a subset of Y with $|Z| = 3$ such that the points of Z are not on the same side of $\operatorname{aff} S_1$. By Lemma 4.5, the dual of the oriented matroid $M := \mathbb{A}\mathrm{ff}(S_1 \cup Z \cup \omega)/\omega$ is isomorphic to the oriented matroid of affine dependences of a subset of $\mathbb{R}^2$ given by one of the drawings of Figure 1. Proceeding as in case 2, we can find two positive circuits S, S' of M such that $|S| = |S'| = 4$ and $|S \cap S'| = 2$. The plane $P := \operatorname{aff} S \cap \operatorname{aff} S'$ contains ω, hence $P = \operatorname{aff} N$ by the perturbation principle. It follows that the two points of $S \cap S'$ belong to $\operatorname{aff} N$: impossible.

Case 4. $k = 4$.

In this last case, 4-positive independence and three-dimensional intersection lead to $|S_i| + |S_j| \geq 9$ for all $i \neq j$, hence $|S_1| = 5$ and a contradiction is immediately obtained. □

8. Proof of Reay's Conjecture for $d = 5$

One key step in the proof of Reay's conjecture in dimension 4 is Theorem 5.2, which characterizes the 4-lopsided oriented matroids of rank 4. It is likely that Conjecture 4.1 also holds for $k = d = 5$ (see the comments in Example 4.10), but the methods of Section 4 would certainly lead to a long distinction of cases. To prove Reay's conjecture in $\mathbb{R}^5$, we choose a slightly different—but still similar—approach, using weaker properties of 5-positively independent oriented matroids but a closer geometric study of the situation.

THEOREM 8.1. *Any set X of $6 \cdot (m-1)+k+1$ points in general position in $\mathbb{R}^5$, with $0 \leq k \leq 5$, is (m,k)-divisible. More precisely, if ω is a Tverberg point of X and if X is only supposed to be* 5*-positively independent at ω, then X is (m,k)-divisible at ω.*

PROOF. As in dimension 4, we proceed by induction on k and m, and we only consider in the proof the case of a subset X of $6 \cdot (m-1)+k+1$ points in $\mathbb{R}^5$, 5-positively independent at ω, with $m \geq 2$ and $1 \leq k \leq 5$. Let $S_1, S_2, \ldots, S_m$ be pairwise disjoint subsets of X of minimal supports such that $N := \bigcap_{i=1}^m \operatorname{conv} S_i$ is at least $(k-1)$-dimensional and contains ω. We again assume ω to be in general position in the relative interior of N and $(|S_1|, |S_2|, \ldots, |S_m|)$ to be lexicographically maximal with respect to these conditions, which implies $6 \geq |S_1| \geq |S_2| \geq \cdots \geq |S_m|$. Finally, we may also assume that $|S_1| < 6$, that N is exactly $(k-1)$-dimensional, and that aff N and $Y := X \setminus \bigcup_{i=1}^m S_i$ are not weakly separated by a hyperplane. We then show that a contradiction can be obtained.

Case 1. $k = 1$.

The preceding assumptions mean that N is reduced to ω, $|S_i| \leq 5$ for all i and $\omega \in \operatorname{conv} Y$, so there is a minimal $(m+1)$th simplex $S_{m+1} \subseteq Y$ such that $\omega \in \operatorname{conv} S_{m+1}$. The lexicographical condition on the S_i shows that $|S_{m+1}| \leq |S_m| \leq 5$ and we clearly have $|S_{m+1}| \geq 2$. In addition, we observe that $|S_p \cup S_q| \geq 7$ for all $p \neq q$, by 5-positive independence. If we had $|S_p \cup S_q| = 8$, then $\mathbb{A}\mathrm{ff}(S_p \cup S_q \cup \omega)/\omega$ would be totally cyclic, 5-lopsided of rank 5, with two disjoint positive circuits, which is impossible by Lemma 4.7. For similar reasons, every positive circuit $S \neq S_p, S_q$ of $\mathbb{A}\mathrm{ff}(S_p \cup S_q \cup \omega)/\omega$ satisfies $(|S_p \cap S|, |S_q \cap S|) = (2,2)$, $(2,3)$ or $(3,2)$. Since $(|S_1|, |S_2|, \ldots, |S_{m+1}|)$ is lexicographically maximal and $m \geq 2$, there are only three possibilities for this $(m+1)$-tuple:

$$(|S_1|, |S_2|, \ldots, |S_{m+1}|) = (5,5,\ldots,5,2), (5,5,\ldots,5,4) \quad \text{or} \quad (5,5,5,\ldots,5).$$

In what follows, we set $E := S_1 \cup S_2$ and $M := \mathbb{A}\mathrm{ff}(E \cup \omega)/\omega$. We shall obtain a contradiction by carrying out a geometric study of the rank 5 acyclic oriented matroid M^*, represented as $\mathbb{A}\mathrm{ff}(\tilde{E})$, where $\tilde{E}$ is a set of 10 (possibly multiple) points of $\mathbb{R}^4$. With the notational conventions of Section 2, S is a positive circuit of E if and only if there is a facet $\tilde{F}$ of $\operatorname{conv} \tilde{E}$ such that $\tilde{S} = \tilde{E} \setminus \tilde{F}$, so the preceding properties have the following translation:

- — $\operatorname{conv} \tilde{E}$ is a four-dimensional polytope;
- — every facet of $\operatorname{conv} \tilde{E}$ contains five or six elements of $\tilde{E}$;
- — any two distinct facets of $\operatorname{conv} \tilde{E}$ have at most three elements of $\tilde{E}$ in common; in particular, every facet of $\operatorname{conv} \tilde{E}$ is a simplicial 3-polytope;
- — $\tilde{F}_1 := \operatorname{conv} \tilde{S}_1$ and $\tilde{F}_2 := \operatorname{conv} \tilde{S}_2$ are facets of $\operatorname{conv} \tilde{E}$;
- — $\tilde{F}_1$ (resp. $\tilde{F}_2$) has four or five vertices; if it has four vertices, the fifth point of $\tilde{S}_1$ (resp. $\tilde{S}_2$) belongs to the relative interior of $\tilde{F}_1$ (resp. $\tilde{F}_2$);
- — every facet $\tilde{F} \neq \tilde{F}_1, \tilde{F}_2$ of $\operatorname{conv} \tilde{E}$ contains five or six points of $\tilde{E}$; more precisely, $(|\tilde{F} \cap \tilde{S}_1|, |\tilde{F} \cap \tilde{S}_2|) = (3,2), (2,3)$ or $(3,3)$, hence each point of $\tilde{F} \cap \tilde{E}$ is an extreme point of $\tilde{F}$.

Let $\tilde{H}$ be a hyperplane that strictly separates $\tilde{S}_1$ and $\tilde{S}_2$, and let $\tilde{P}$ denote the 3-polytope $\tilde{H} \cap \operatorname{conv} \tilde{E}$. Each facet $\tilde{F}'$ of $\tilde{P}$ is obtained as the intersection of $\tilde{H}$ with a facet $\tilde{F} \neq \tilde{F}_1, \tilde{F}_2$

of conv $\tilde{E}$. Since $\tilde{F}$ is simplicial, $\tilde{H}$ intersects five edges of $\tilde{F}$ if $(|\tilde{F} \cap \tilde{F}_1|, |\tilde{F} \cap \tilde{F}_2|) = (3, 2)$ or $(2, 3)$, and six edges of $\tilde{F}$ if $(|\tilde{F} \cap \tilde{F}_1|, |\tilde{F} \cap \tilde{F}_2|) = (3, 3)$, hence each face $\tilde{F}'$ of $\tilde{P}$ is either a pentagon or a hexagon. On the other hand, $\tilde{F} \cap \tilde{F}_1$ (resp. $\tilde{F} \cap \tilde{F}_2$) is a triangular face of $\tilde{F}_1$ (resp. of $\tilde{F}_2$) if $|\tilde{F} \cap \tilde{F}_1| = 3$ (resp. if $|\tilde{F} \cap \tilde{F}_2| = 3$) and, conversely, every triangular face of $\tilde{F}_1$ (resp. of $\tilde{F}_2$) is contained in a unique facet $\tilde{F} \neq \tilde{F}_1, \tilde{F}_2$ of conv $\tilde{E}$. Since $\tilde{F}_1$ and $\tilde{F}_2$ have at most six triangles, it follows that $\tilde{P}$ has at most 12 faces. How about: under these conditions, only one possibility now remains:

— $\tilde{P}$ is isomorphic to a regular dodecahedron (the only 3-polytope with at most 12 faces, all of which have at least five vertices);
— each facet of conv $\tilde{E}$ is isomorphic to a bipyramid (the only simplicial 3-polytope with five vertices).

Next, let us consider an edge $[\tilde{a}, \tilde{b}]$ of $\tilde{F}_1$ such that $[\tilde{a}, \tilde{b}] = \tilde{F} \cap \tilde{F}_1$, where $\tilde{F}$ is a facet of conv $\tilde{E}$. Then, $[\tilde{a}, \tilde{b}]$ belongs to at least four facets of conv $\tilde{E}$, i.e., $M(E \setminus \{a, b\})$ has at least four positive circuits. On the other hand, $M(E \setminus \{a, b\})$ cannot have more than four positive circuits by Lemma 4.7, hence $\tilde{F}$ is the only facet of conv $\tilde{E}$ such that $[\tilde{a}, \tilde{b}] = \tilde{F} \cap \tilde{F}_1$. It follows that $\tilde{F}_1$ has exactly six edges of this type, and (at least) three of them must be incident to some vertex $\tilde{x}$ of $\tilde{F}_1$. To conclude, we observe that $\tilde{x}$ then belongs to at least seven facets of conv $\tilde{E}$, which contradicts Lemma 4.8 applied to $M(E \setminus x)$.

Case 2. $k = 2$.

5-positive independence and one-dimensional intersection imply $|S_i| + |S_j| \geq 8$ for $i \neq j$, hence $|S_i| = 4$ or 5 for all i. Since aff N and Y are not weakly separated by a hyperplane, we have $|Y| \geq 5$. Moreover, by considering the hyperplane aff S_1 (or aff$(S_1 \cup y)$ with $y \in Y$ if $|S_1| = 4$), we can find a subset Z of Y such that $|S_1 \cup Z| = 9$ and ω is interior to conv $(S_1 \cup Z)$. The oriented matroid $M := \mathbb{A}\mathrm{ff}(S_1 \cup Z \cup \omega)/\omega$ is then realizable, 5-lopsided, totally cyclic of rank 5, hence M^* is represented as $\mathbb{A}\mathrm{ff}(\tilde{E})$, where $\tilde{E}$ is one of the subsets of $\mathbb{R}^3$ described in Lemma 4.5. Since some face of conv $\tilde{E}$ contains four points of $\tilde{E}$, the lexicographical condition on the S_i implies that, in fact, $|S_1| = 5$. Moreover, the face conv $\tilde{Z}$ of conv $\tilde{E}$ is adjacent to at least three other faces, that we denote by $\tilde{F}$, $\tilde{F}'$ and $\tilde{F}''$, the notation being chosen so that $\tilde{F}$ is also adjacent to both $\tilde{F}'$ and $\tilde{F}''$.

2.1. conv $\tilde{E}$ has an interior point.

Setting $\tilde{a}$ to be the interior point of conv $\tilde{E}$, the positive circuits $S := E \setminus F$ and $S' := E \setminus F'$ of M satisfy $|S| = |S'| = 5$ and $S \cap S' = \{a, b, z\}$, with $b \in S_1$ and $z \in Z$. The hyperplanes aff S and aff S' of $\mathbb{R}^5$ contain a, b, z and ω, hence a, b, z and aff N (by the perturbation principle) are distinct by Lemma 3.4. Their intersection is thus a 3-space P containing a, b, z and aff N. The situation where $P = \mathrm{aff}(N \cup \{a, b\})$ is impossible: it would imply $z \in \mathrm{aff}\, S_1$, contradicting Lemma 3.4. It follows that a, b and aff N are coplanar. Restarting the argument with $\tilde{F}$ and $\tilde{F}''$ yields a point $c \in S_1$ (with $c \neq b$) such that a, c and aff N are also coplanar. Then, the points a, b, c and ω are affinely dependent, which leads to a contradiction since $\{a, b, c\}$ is strictly included in a circuit of M.

2.2. conv $\tilde{E}$ has no interior point.

The method is similar, and in fact slightly easier since we can find positive circuits S and S' such that $S \cap S' = \{a, b, c, z\}$, with $a, b, c \in S_1$ and $z \in Z$.

Case 3. $k = 3$.

Reasoning as in case 2, we have $|S_1| = 5$ and $|Y| \geq 4$. The same method applies since we can again find a subset Z of Y such that $|Z| = 4$ and ω is interior to conv $(S_1 \cup Z)$.

Case 4. $k = 4$.

We also have $|S_1| = 5$ and we can find a subset Z of Y such that $|Z| = 3$ and ω is interior to conv $(S_1 \cup Z)$. Using the representation given by Lemma 4.7, there exist positive circuits S and S' of M such that $|S \cap S'| = 3$. Then, aff $S \cap$ aff S' is a 3-space containing ω, hence aff N, and we deduce that aff $S \cap$ aff $S' =$ aff N contains the three elements of $S \cap S'$: impossible.

Case 5. $k = 5$.

In this last case, 5-positive independence and four-dimensional intersection directly lead to $|S_1| = 6$. □

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